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Steady-State Harmonic Analysis of Phase Shift Oscillators

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Abstract **– This paper describes an improved method for calculating the steady-state oscillation frequency and fundamental and harmonic amplitudes of phase shift oscillators (transient behaviour is *not* considered). The method is quite general and although the examples described in the paper relate to oscillators with *RC* phase shift networks, the principles apply equally to *LCR*-based oscillators. The method described proposes a novel extension to the method of *harmonic balance* that includes harmonics in the main feedback loop in addition to the fundamental. This is significant because it allows a more accurate calculation of the oscillation frequency, fundamental and harmonic amplitudes in oscillators with higher signal levels, less selective phase shift networks and/or higher levels of nonlinearity than is possible with existing methods. This development leads to a more effective design process and hence simpler, more economic circuits. New closed form expressions for the main design parameters are presented.**

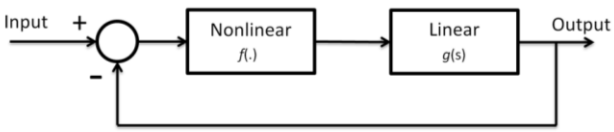
Keywords **– Phase-Shift Oscillators, Harmonic Balance, Sine Waves With Low Harmonic Distortion, Circuit Analysis.**

I. Introduction

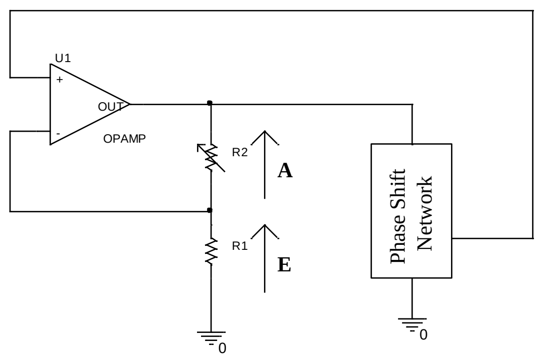
The economical generation of harmonically pure sine-waves, particularly in integrated circuit form, remains an important problem in circuit design, having applications in a wide variety of systems. At lower frequencies, up to about 100 kHz, *RC*-active circuits are frequently employed for this purpose and a number of suitable circuits have been described [1]-[3]. At higher frequencies it is usually necessary to resort to *LC* phase shift circuitry and although the methods developed below apply *in principle* to such circuits, they are not pursued in this paper. *RC*-active designs use operational amplifiers (op-amps) as gain blocks or, in some more recent designs, operational transconductance amplifiers (OTAs) [4], [5]. The inherently wider bandwidth of OTAs makes them suitable for sine wave generation at higher frequencies and confers other advantages as well, for example the possibility of frequency control by means of a single resistor (see [6], [7] and surrounding references), or OTA transconductance [5].

All classical sine wave oscillators consist of two main sections: (a) a linear frequency-defining (phase-shift) network *g*(s) (which includes linear amplification) and (b) a memory-less non-linear section *f*(.) to define and stabilise the amplitude of the signal (see [4] and its extensive list of references). Two main types of general non-linear block (b) have been proposed. These are (i) a limiter placed in the signal path and (ii) some form of automatic gain control loop that is separate from and additional to the signal path. Circuits of type (i) are generally simpler and more economical than those of type (ii), but the presence of nonlinear elements in the signal path normally results in higher levels of harmonic distortion. In this paper the emphasis is on type (i) nonlinearities although the methods presented are in fact quite general. Fig 1(a) gives the block diagram of a feedback system in which the two sections are connected in cascade and overall feedback is applied (the input is zero for self-sustaining limit cycle behaviour). Fig 1(b) shows a practical realisation of a sine wave oscillator in which g(s) is a three terminal phase shift network while *f*(.) is a non-linear resistor connected in the feedback loop of a non-inverting operational amplifier circuit [8]-[12].

Once steady state limit cycle (i.e. periodic) behaviour has been established, it is possible to investigate the resulting signal in the frequency domain using the *describing function method* (also called the method of *harmonic balance*) [12]. In this approach the first harmonic (fundamental) only is used to solve the feedback equations and using Fourier series analysis it is then possible to predict the harmonic content of the resulting signal. The method is especially effective when *g*(s) is a highly selective bandpass filter as is frequently the case for *LC* phase shift networks. In these oscillators the magnitude of *g*(s) falls away rapidly either side of the resonant frequency suppressing oscillation at frequencies other than the resonant frequency of *g*(s). However, for *RC* phase shift networks, such as the much used Wien bridge circuit, this is not the case and the harmonic components are much harder to control. In addition to contributing distortion to the output, the harmonic components cause shifts in both the frequency and amplitude of the resulting sinusoid.



(a)



(b)

Fig.1. (a) Basic feedback system and (b) realisation of a general phase shift oscillator employing a single operational amplifier

This paper describes a developed version of the method of harmonic balance to determine the *steady-state* behaviour of *RC* phase shift oscillators employing quadratic or cubic type-(i) (limiting) nonlinearities resulting in the generation of *odd*-order harmonics only. The method shows significantly improved accuracy compared to the simple method described above (i.e. where only the fundamental is fed back). Closed form expressions are provided for *four* parameters for the oscillator: the frequency and amplitude of the fundamental, the amplitude of the 3rd harmonic and the phase angle between the fundamental and harmonic. Since the nonlinear polynomial function is constructed from the linear superposition of quadratic and cubic elements whose Fourier series expansions are known, the method lends itself to a flexible 'building block'-type approach. If yet more precision is required, the method is readily extendable to cases where more than one harmonic is fed back, although this more complex formulation is not discussed in detail.

Section II outlines the mathematical basis of the method and the types of nonlinearity to be considered. Section III describes the improved formulation, while Section IV contains simulation results from several examples. Section V contains discussion material and, finally, the appendix deals in some detail with the Fourier methods employed in the paper which, though elementary in themselves, are fundamental to the methods described.

II. Analysis Method

Fig 1(b) shows the schematic of a generalised RC-active oscillator using an op-amp with a limiting nonlinearity represented by the nonlinear resistor R_2 (the analysis also applies to OTA-based systems without loss of generality). The phase-shift network is usually a second-order system so that when connected in a positive feedback loop its poles can be placed in the right half of the complex plane to ensure start-up [9]-[12]. The phase-shift network can be of lowpass, highpass or bandpass type. The gain of the amplifier, which is assumed to be ideal (i.e, no phase shift- see ref [1]), is controlled by the ratio of R_2/R_1 where the voltages $\underline{A}$ and $\underline{E}$ are complex quantities. As already noted, for the purposes of this paper, R_2 realises a *cubic* or *quadratic* nonlinearity, or a weighted sum of both, plus a linear term.

For the *quadratic* case, the large signal conductance G (= $1/R_2$) is described by the following polynomial expression:

$$G(\underline{A}) = G_0 + \beta|\underline{A}| \tag{1a}$$

where $\underline{A}$ is the voltage across R_2, β is a scaling parameter with dimension IV^{-2} and G_0 is a constant conductance. The current flowing in R_2 is therefore:

$$I = G\underline{A} = G_0\underline{A} + \beta|\underline{A}|\underline{A} \tag{2a}$$

For the *cubic* case, the conductance is given by the following expression:

$$\begin{aligned} G(\underline{A}) &= G_0 + \alpha|\underline{A}|^2 \\ &= G_0 + \alpha\underline{A}^2 \end{aligned} \tag{1b}$$

where α is a scaling parameter and the current is therefore:

$$\begin{aligned} I = G\underline{A} &= G_0\underline{A} + \alpha|\underline{A}|^2\underline{A} \\ &= G_0\underline{A} + \alpha\underline{A}^3 \end{aligned} \tag{2b}$$

Eqns (1) and (2) can be combined (with suitable weights) to form a general 3rd order polynomial to represent the nonlinear output current. This is discussed in Section III(b).

For steady-state oscillation to occur, as already noted, the poles of the system must lie on the imaginary axis of the complex plane. This requires (i) unity loop gain and (ii) zero phase-shift at the fundamental frequency, the so-called *Barkhausen Criterion* for oscillation [8]. However, due to the feedback nature of the circuit, the loop also contains steady-state harmonic components. For the nonlinearities considered in this paper, these are restricted to odd-order multiples of the fundamental frequency. The presence of harmonics in the feedback loop has a significant effect on the behaviour of the circuit including the amplitudes of the harmonic components and the oscillation frequency [13]. The oscillator is described by the following system of simultaneous complex (harmonic balance) equations obtained from analysis of Fig 1 where $\underline{H}$ is the complex transfer function of the phase shift network:

$$\begin{aligned} &\underline{H}_1(\underline{A}_1 + \underline{E}_1)\sin\omega t = \underline{E}_1\sin\omega t \\ &\underline{H}_2(\underline{A}_2 + \underline{E}_2)\sin 3\omega t = \underline{E}_2\sin 3\omega t \\ &\vdots \\ &\underline{H}_n(\underline{A}_n + \underline{E}_n)\sin n\omega t = \underline{E}_n\sin n\omega t \end{aligned}$$

where $\underline{A}_i$ and $\underline{E}_i$ are complex voltages and n is the highest order harmonic considered in the analysis. The equations (3) can be written (omitting the explicit time dependence):

$$\underline{A}_i = \left(\frac{1-\underline{H}_i}{\underline{H}_i}\right)\underline{E}_i \text{ ; all } i \text{ odd}$$

Inverting:

$$\underline{E}_i = \left(\frac{\underline{H}_i}{1-\underline{H}_i}\right)\underline{A}_i = \underline{J}_i\,\underline{A}_i \tag{4}$$

In order to solve equation (4), a further relationship between $\underline{A}_i$ and $\underline{E}_i$ is required. Since $\underline{E}_i$ is a scaled version of the output current of G, if the Fourier components of I in eqn (2a) are known then $\underline{A}_i$ and $\underline{E}_i$ can be calculated and a steady-state solution obtained. This is the essence of the *harmonic balance* method and the process is described further in the Appendix. For a single steady-state sinusoidal input of amplitude A and angular frequency ω, the output current *I(t)* is given by the Fourier series expansion of eqn (2a/b) [14]-[15]. For both quadratic (eqn(2a)) and cubic cases (eqn(2b)) the resulting expansion consists of odd-order *sin* terms only.

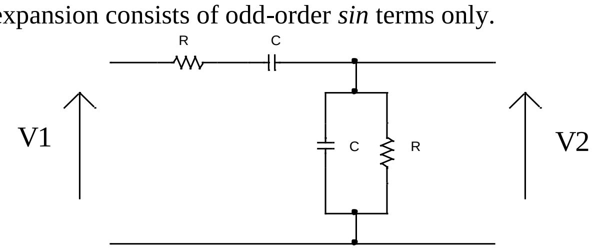


Fig.2. *Wien* bandpass phase shift network

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So, e.g, for the quadratic case:

$$I(t) = G_0 A \ \sin\omega t + \beta A^2[C_1 \sin\omega t + C_3 \sin 3\omega t \ldots] = G_0 A \ \sin\omega t + \beta[F_1 \sin\omega t + F_3 \sin 3\omega t \ldots] \quad (5)$$

where:

$F_i = A^2 C_i$ all i, and

$$I(t) = I_1 \sin\omega t + I_3 \sin 3\omega t + \ldots + I_n \sin n\omega t$$

equating the coefficients of the harmonic components:

$$I_1(t) = G_0 A_1 + \beta A_1^2 C_1$$
$$I_3(t) = \beta A_1^2 C_3 \quad (6)$$
$$\vdots$$
$$I_n(t) = \beta A_1^2 C_n$$

where the *odd*-order Fourier coefficients C_i are given in the Appendix. A first-order solution of eqn (4), using the fundamental and ignoring the harmonics, is now possible for a particular phase shift network, e.g., the *Wien* bandpass network shown in Fig 2. From equations (2a) and (6) we have:

$$I(t) = [G_0 A_1 + \beta A_1^2 (0.85)] \sin\omega t \quad (7)$$

By analysis of Fig 2, at resonance, $\omega CR = 1$ and so $\underline{H}_1(j\omega) = 1/3 + j0$ and from eqn (4) $\underline{J}_1(j\omega) = 1/2 + j0$. Table I gives the values of $\underline{H}$ and $\underline{J}$ for the *Wien* phase shift network. From eqns (4) and (7) and the value of $\underline{J}_1$ in Table I:

$$A_1 = 2E_1$$
$$= 2R_1 I_1 = 2R_1[G_0 A_1 + \beta A_1^2 C_1]$$

Hence, inverting:

$$A_1 = \frac{1 - 2G_0 R_1}{\beta R_1 C_1} \quad (8a)$$

The numerator of eqn (8a) gives the limiting condition for oscillation:

$$G_0 R_1 \leq 0.5$$

E.g., if β = 5e-4, R_1 = 1e3 and G_0 = 4.575e-4, then: A_1 = 0.1 volts (peak). In order to calculate the fundamental output voltage, V_1, we note from eqn. (4) the following *output equation*:

$$\underline{V}_i = \underline{E}_i + \underline{A}_i = (1 + \underline{J}_i)\underline{A}_i$$
$$= 1.5A_1 = 0.15 \text{ volts.}$$

For the *cubic* nonlinearity, exactly the same principles apply, with the corresponding value for A_1 given by:

$$A_1 = \sqrt{\frac{1 - 2G_0 R_1}{\alpha R_1 C_1}} \quad (8b)$$

Note also that in this simple first-order model, the oscillation frequency f_0 is determined entirely by the resonant frequency of the phase-shift network, i.e, the frequency at which its phase shift is zero:

$$f_0 = \frac{1}{2\pi CR}$$

So if C = 120 pF and R = 50 kΩ, f_0 = 26.52 kHz.

Returning to the case of the *quadratic* nonlinearity (although again exactly similar remarks apply to the cubic case), in order to estimate the harmonic content of the signal generated by the oscillator, assume initially that the signal amplitude is sufficiently small that the output of the nonlinearity can be expressed as the Fourier expansion of its response to a pure sine wave at the fundamental frequency. This is the approximate approach described in [4]. Hence we can calculate the ratio of E_3 to E_1. E.g., employing eqns (5-7) for the 3rd harmonic we have:

$$\frac{E_3}{E_1} = \frac{\beta A_1 C_3}{G_0 + \beta A_1 C_1}$$

And hence, using (4) it can be shown that:

$$\frac{V_3}{V_1} = \frac{\beta A_1 C_3}{G_0 + \beta A_1 C_1} \cdot \frac{H_1}{H_3}$$

Table II compares these calculated results, including the oscillation frequency, to PSPICE simulations. The table shows that although the amplitude of the fundamental component can be accurately calculated using this method, the calculated values for the harmonics and the oscillation frequency are much less precise, especially for larger signal amplitudes. The main weakness in the method is the assumption that the harmonic content of the oscillator output signal can be calculated from a single pure sinusoidal input to the non-linear block (i.e, the fundamental). This is only applicable in cases where most of the terms of the Fourier series expansion can be ignored. The results tabulated above show that for larger signal amplitudes the presence of harmonics in the feedback loop (and hence at the *input* to the nonlinearity) significantly influence both the output harmonic components and the oscillation frequency.

A more sophisticated approach considers the response of G when driven not only by a fundamental sinusoid, as assumed in [4], but also by the harmonics of the sinusoid, fed back and filtered by the action of the phase shift network. In this more complete formulation, the harmonic components are permitted to have arbitrary amplitudes and phase relationships to the fundamental. This input/output relationship is considered in the next section.

III. Extended Harmonic Balance Analysis

(a) Analysis method for quadratic and cubic nonlinearities

In order to allow the nonlinear block to have inputs at harmonic frequencies as well as at the fundamental, the formulism described in the previous section needs to be extended. In this paper we represent the nonlinearity as a time-invariant linear system with inputs and outputs truncated to the first two terms of a Fourier series, i.e, the fundamental and third harmonic. The method is indicated schematically in Fig 3 where the nonlinear conductance G is driven by a voltage that is the weighted sum of a fundamental sinusoid and a further sinusoid at the third harmonic, with an arbitrary phase angle φ between the two. The output current spectrum is truncated to a similar format, the output phase angles being denoted Φ_i. This allows the steady-state relationship between the input and output to be determined. Note that the phase of the input at the fundamental is taken as zero: this is treated as the phase reference point for the whole system. With reasonable assumptions, it can be shown that for the *quadratic* nonlinearity (see Appendix):

$$\underline{F}_1 = 0.85A_1^2 - j0.51A_1A_3\sin\phi$$
$$\underline{F}_3 = -0.17A_1^2 + 1.31A_1A_3\cos\phi + j1.24A_1A_3\sin\phi \qquad (9a)$$

where the Fourier coefficients $\underline{F_i}$ are now complex quantities as well as being functions of both inputs A_1 and A_3 and also φ (note that these equations reduce to the form of eqn (6) if A_3 is set to zero). Similarly, for the *cubic* nonlinearity we have:

$$\underline{F}_1 = 0.75A_1^3 - j0.75{A_1}^2A_3\sin\phi$$
$$\underline{F}_3 = -0.25A_1^3 + 1.50{A_1}^2A_3\cos\phi + j1.50{A_1}^2A_3\sin\phi \qquad (9b)$$

The fact that the first equation of each pair (i.e, 9a and 9b) is complex means that there is a non-zero phase angle (Φ_1) between the input and output at the fundamental frequency that is typically about 1°, even if the amplifier is ideal [1]. In order for the Barkhausen criterion to be satisfied for oscillation to occur, there must be a compensating angle in the phase shift network. The effect of this is that the system does not operate as a 'classical' phase shift oscillator with gain 1/3 and zero phase angle (as described in Section II), but with second- order offsets in both parameters and also the oscillation frequency. These offsets were ignored in the simplified analysis presented in [13] and first-order 'working' versions of the functions (9a & b) were adopted as follows for the quadratic and cubic

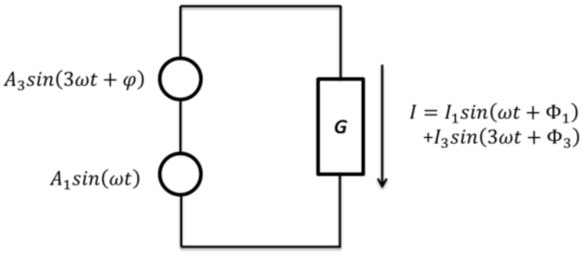

Fig.3. Format for representing the steady-state current in the conductance G when a harmonic is included.

nonlinearities respectively:

$$\underline{F}_1 = 0.85A_1^2$$
$$\underline{F}_3 = -0.17A_1^2 + 1.31A_1A_3\cos\phi + j1.24A_1A_3\sin\phi \qquad (10a)$$
$$\underline{F}_1 = 0.75A_1^3$$
$$\underline{F}_3 = -0.25A_1^3 + 1.50{A_1}^2A_3\cos\phi + j1.50{A_1}^2A_3\sin\phi \qquad (10b)$$

The full expressions can be used to improve the design process. For the quadratic case, using the full (complex) form of eqn (9a), the total current output from G is given by:

$$\underline{I}(t) = [G_0\underline{A}_1 + \beta\underline{F}_1]\sin\omega t + [G_0\underline{A}_3 + \beta\underline{F}_3]\sin 3\omega t \qquad (11)$$

For the *fundamental* component, using eqn (4) and the complex form of the Fourier coefficients (eqn 9a) and omitting the explicit time dependence:

$$\underline{E}_1 = \underline{I}_1R_1$$
$$= [G_0\underline{A}_1 + \beta\underline{F}_1]R_1 = G_0R_1\underline{A}_1 + \beta R_1(0.85A_1^2 - j0.51A_1A_3\sin\phi)$$
$$= \underline{J}_1\,\underline{A}_1 \qquad (12)$$

For the particular case of the *Wien* bandpass phase shift network:

$$\underline{J}_1 = \frac{j\omega CR}{(1-\omega^2C^2R^2) + j2\omega CR} = \frac{jx}{(1-x^2) + j2x}$$
$$= \frac{2x^2 - jx(1-x^2)}{(1+x^2)^2} \qquad (13)$$

where $x = \omega_0 CR$ ($x = 1$ only in the ideal case) and so:

$$f_0 = \frac{\omega_0}{2\pi} = \frac{x}{2\pi CR}$$

Note also that $\underline{A_1} = A_1$, since as already noted this A_1 is taken as the zero phase reference for the system. Combining eqns (12) and (13) and equating real and imaginary parts:

$$R_1(G_0 + \beta 0.85A_1) = \frac{2x^2}{(1+x^2)^2} \qquad (14)$$

$$-0.51\beta R_1A_3\sin\phi = \frac{-x(1-x^2)}{(1+x^2)^2} \qquad (15)$$

Note that if $x = 1$, eqn (14) reduces to eqn (8a) which is the same as using the simplified 'working' versions of the Fourier coefficients (eqns 10a, b) and the right hand side of eqn (15) is zero. This implies a fundamental inconsistency in the simplified theory presented in [13] since it requires A_3 and/or φ to be zero. Neither proposition is true. On the other hand, knowledge of A_3 and φ enables x to be found and hence a better approximation to the value of f_0 to be made.

For the *third harmonic* we can write (by analogy with eqn (13)):

$$\underline{E}_3 = \underline{I}_3R_1$$
$$= [G_0\underline{A}_3 + \beta\underline{F}_3]R_1$$
$$= G_0R_1A_3(\cos\varphi + j\sin\varphi) + \beta R(-0.17A_1^2 + 1.31A_1A_3\cos\varphi + j1.24A_1A_3\sin\varphi)$$
$$= \underline{J}_3\,\underline{A}_3 \qquad (16)$$

where the 3rd harmonic component:

$$\underline{A}_3 = A_3\sin(3\omega t + \phi)$$

is written in the form:

$$A_3\sin 3\omega t(\cos\phi + j\sin\phi)$$

Proceeding as for the fundamental and noting that:

$$\underline{J}_3 = \frac{j3\omega CR}{(1-9\omega^2C^2R^2) + j6\omega CR} = \frac{j3x}{(1-9x^2) + j6x}$$
$$= \frac{18x^2 - j3x(1-9x^2)}{(1+9x^2)^2}$$

we can show that:

$$\underline{J}_3\,\underline{A}_3 = \frac{[18x^2\cos\varphi - 3x(1-9x^2)\sin\varphi] - j[3x(1-9x^2)\cos\varphi + 18x^2\sin\varphi]}{(1+9x^2)^2}$$

Substituting for $\underline{J_3}\,\underline{A_3}$ in eqn (16) and equating real and imaginary parts:

$$G_0R_1A_3\cos\phi + \beta R(-0.17A_1^2 + 1.31A_1A_3\cos\phi) = \frac{\left[18x^2\cos\phi + 3x\left(1-9x^2\right)\sin\phi\right]}{\left(1+9x^2\right)^2}.A_3 \quad (17)$$

$$(G_0R_1 + 1.24\beta RA_1)A_3\sin\varphi = \frac{\left[-3x\left(1-9x^2\right)\cos\varphi + 18x^2\sin\varphi\right]}{\left(1+9x^2\right)^2}.A_3 \quad (18)$$

Considering the quartet of eqns (14 - 17), we note that for x close to 1, the sensitivity of all but eqn (15) to the actual value of x is *low*. It is reasonable, therefore, to set x = 1 throughout the set except in the term where $(1-x^2)$ appears on the right hand side of eqn (15):

$$R_1(G_0 + \beta 0.85A_1) = 0.5 \quad (19a)$$

$$R_1 0.51\beta A_3\sin\phi = 0.25\left(1-x^2\right) \quad (19b)$$

$$R_1[G_0A_3\cos\phi + \beta(-0.17A_1^2 + 1.31A_1A_3\cos\phi)] = \left(0.18\cos\phi - 0.24\sin\phi\right)A_3 \quad (19c)$$

$$R_1(G_0 + 1.24\beta A_1)A_3\sin\phi = \left(0.24\cos\phi + 0.18\sin\phi\right)A_3 \quad (19d)$$

In order to solve this set of simplified equations, we note immediately that the criterion for oscillation and the expression for the amplitude of A_1 (eqn (19a)) is exactly as in the simple case (see eqn (8a)). Secondly, since A_3 cancels from eqn (19d), we can calculate φ:

$$\phi = -\tan^{-1}\left(\frac{0.24}{G_0R_1 + 1.24\beta RA_1 - 0.18}\right) \quad (20)$$

Substituting for A_1 from eqn (8a/19a):

$$\phi = -\tan^{-1}\left(\frac{0.24}{0.46G_0R_1 - 0.55}\right) \quad (21)$$

Thirdly, substituting A_1 and φ in eqn (19c) enables A_3 to calculated which finally allows x (and hence f_0) to calculated by substitution in eqn (19b).

Exactly similar principles apply in the case of a *cubic* nonlinearity and, using the same assumption about $x \approx 1$ as in the quadratic case we derive the following equation set:

$$R_1(G_0 + \alpha 0.75A_1^2) = 0.5 \quad (22a)$$

$$R_1 0.75\alpha A_1A_3\sin\phi = 0.25\left(1-x^2\right) \quad (22b)$$

$$R_1[G_0A_3\cos\phi + \alpha(-0.25A_1^3 + 1.50A_1^2A_3\cos\phi)] = \left(0.18\cos\phi - 0.24\sin\phi\right)A_3 \quad (22c)$$

$$R_1(G_0 + 1.50\alpha A_1^2)A_3\sin\phi = \left(0.24\cos\phi + 0.18\sin\phi\right)A_3 \quad (22d)$$

which can be solved using the same procedure as for the *quadratic* case.

(b) Extension to a general 3rd order polynomial nonlinearity

Consider a general 3rd order nonlinear polynomial function of the form:

$$G(\underline{A}) = G_0 + \beta|\underline{A}| + \alpha|\underline{A}|^2 \quad (23)$$

with the corresponding output current:

$$I = G\underline{A} = G_0\underline{A} + \beta|\underline{A}|\underline{A} + \alpha|\underline{A}|^2\underline{A} = G_0\underline{A} + \beta|\underline{A}|\underline{A} + \alpha\underline{A}^3 \quad (24)$$

Using the methods developed above for the quadratic and cubic cases, the following equation set can be readily derived for the nonlinearity of eqn (23) for the *Wien* bandpass phase-shift network:

$$R_1(G_0 + \beta 0.85A_1 + \alpha 0.75A_1^2) = 0.5 \quad (25a)$$

$$R_1(0.51\beta + 0.75\alpha A_1)A_3\sin\phi = 0.25\left(1-x^2\right) \quad (25b)$$

$$R_1[G_0A_3\cos\phi + \beta(-0.17A_1^2 + 1.31A_1A_3\cos\phi) + \alpha(-0.25A_1^3 + 1.50A_1^2A_3\cos\phi)] = \left(0.18\cos\phi - 0.24\sin\phi\right)A_3 \quad (25c)$$

$$R_1(G_0 + 1.24\beta A_1 + 1.50\alpha A_1^2)A_3\sin\phi = \left(0.24\cos\phi + 0.18\sin\phi\right)A_3 \quad (25d)$$

The equations can be solved using the same methods as for the eqn sets (19, 22). Beginning with eqn (25a), solving the quadratic for A_1:

$$A_1 = \frac{-0.85\beta \pm \sqrt{0.72\beta^2 + 3\alpha\left(0.5 - G_0R_1\right)}}{1.5\alpha} \quad (26)$$

with the resulting criterion for oscillation:

$$0.5 - G_0R_1 \geq 0$$

(which is the same as in the single quadratic and cubic cases). The next step is to derive φ from eqn (25d):

$$\phi = -\tan^{-1}\left(\frac{0.24}{G_0R_1 + R_1(1.24\beta A_1 + 1.50\alpha A_1^2) - 0.18}\right)$$

and the expression for A_1 can be substituted from eqn (26) as in the cubic and quadratic cases. Having calculated φ, the values for A_3 and x follow from eqns (25c) and (25b), completing the solution.

IV. Simulation Examples

1. Wien bridge Examples with quadratic nonlinearity

These repeat the example quoted above in Section II, but employ the methods developed in Section III for the 3rd harmonic and the oscillation frequency. Also a third case is considered with G_0 = 2.5e-4 (In all cases, as before, C = 120 pF, R = 50 kΩ, β = 5e-4 AV^{-2}, R_1 = 1 kΩ). The results are summarised in Table III and are clearly much more satisfactory than those presented in Table II although φ is somewhat less accurately predicted than the other parameters.

2. Highpass ladder examples

The highpass ladder phase shift network used for these examples is shown in Fig 4. It has the following transfer function:

$$H(x) = \frac{x^3}{j\left(1-6x^2\right) + x\left(x^2-5\right)}$$

where x = ωCR. For zero phase shift (resonance), we require that:

$$x = 1/\sqrt{6}$$

At which frequency H = -1/29 and the phase shift is 180º. In order to sustain oscillation, an inversion is required, so:

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$$H(x) = \frac{-x^3}{j(1-6x^2) + x(x^2-5)}$$

and:

$$J(x) = \frac{-x^3}{j(1-6x^2) + x(2x^2-5)}$$

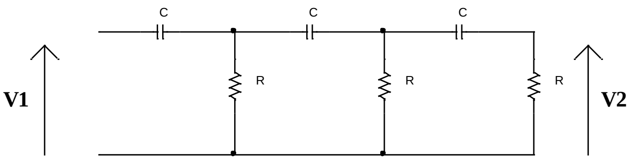

Fig.4. Highpass *ladder* phase shift network

$$= \frac{-x^3\left[x(2x^2-5) + j(6x^2-1)\right]}{(1-6x^2)^2 + x^2(2x^2-5)^2}$$

so that, setting $x = 1/\sqrt{6}$:

$$J_1 \cong 3.57e^{-2} - j1.87e^{-2}(6x^2-1)$$

and:

$$J_3 \cong 6.43e^{-2} - j0.21$$

It is now possible to construct and solve an equation set of the form of eqn (19a-d), for the quadratic nonlinearity. Two cases are considered and the results are summarised in Table IV. As in the case of the *Wien bridge*, the method predicts the four major parameters of the oscillator with good accuracy.

V. Discussion

The proposed analysis method employs a developed version of the *method of harmonic balance* in which the first harmonic is fed back in addition to the fundamental, as is conventional in the design and characterisation of phase-shift oscillators. Together with a linear phase-shift network, this formulation of the effect of general non-linear elements (expressed as polynomial functions) allows the steady-state behaviour of the oscillator to be calculated more accurately than is possible using the simpler methods most commonly employed. The method does *not*, however, enable calculation of the transient response or start-up condition. Some Fourier analysis is required but has only to be carried out once for each order of nonlinearity and order of truncation and a library of representative functions can be readily compiled. Using these standard functions as ‘building blocks’, nonlinear polynomial functions of arbitrary order can be constructed.

(a) Application of the method

A very common form of nonlinearity employed in phase shift oscillators consists of a cubic function plus a linear term. This has the advantage of allowing the design problem to be framed analytically as a realisation of the Van der Pol equation familiar from nonlinear dynamics and for which a large literature exists (see [9]-[11] and surrounding references). There are many ways of realising such a nonlinear function using, e.g, MOS transistors, bipolar transistors and CMOS OTAs biased in the *subthreshold* region of operation [5]. A pair of back-to-back MOS diodes biased close to threshold displays a conductance with near cubic nonlinearity for small signals (< 50 mV) which tends to quadratic form for larger signals. At intermediate signal levels, a better fit is provided by a general 3rd order polynomial of the form of eqn (23).

(b) Limitations of the method

As already stated, the proposed method cannot predict the *transient* behaviour of the system. In addition, although it predicts the oscillation frequency f_0 and the magnitudes of the fundamental and 3rd harmonic signal amplitudes A_1 and A_3 to a good level of accuracy, this is less true for the phase angle between them (φ). Furthermore, given the level of truncation discussed above (2 inputs, 2 outputs) it is not possible to predict higher harmonics with any precision. Consider the case of a *Wien* bridge oscillator with quadratic nonlinearity: φ is given by eqn (21) and since, from eqn (8a), for oscillation, we require that:

$$0 \leq G_0 R_1 \leq 0.5$$

and we can calculate the corresponding range of allowable values of φ:

$$24° \geq \varphi \geq 37°$$

This is quite a restricted range suggesting that A_1 and A_3 are relatively insensitive to the actual value of φ. The reason for the low accuracy in the calculation of φ is that the polynomial G was truncated to two terms only-the fundamental and 3rd harmonic. A characteristic of the *Wien* bandpass phase-shift network and other 2nd-order sections is that the harmonic components decrease quite slowly with frequency and so the coefficient A_5 is probably significant in the calculation of φ, if not A_3. In order to improve this situation the format for calculating eqn (A1) could be modified to include A_5 and, of course, an extra phase term φ_5:

$$f(t) = \left[A_1 \sin\omega t + A_3 \sin(3\omega t + \phi_3) + A_5 \sin(5\omega t + \phi_5)\right]^3$$

for the *cubic* nonlinearity. This will enable φ_3 (ie, φ in the notation in this paper) to be calculated accurately, but not φ_5 which would require yet more terms and so on.

Finally, it is worth re-iterating that throughout the paper we have assumed that the amplifier is ideal in the sense that its bandwidth is infinite and so it contributes no frequency-dependent phase distortion. Of course, this cannot be true in practice, except perhaps for operation at very low frequencies. However, the effect of the presence of amplifier poles at finite frequencies has been examined extensively in the literature (see [1] and surrounding references) and although it would be a relatively simple matter to incorporate such effects in the above analysis, it was felt that such additional analysis is beyond the scope of this paper.

VI. Conclusion

This paper has described an improved method for calculating the steady-state oscillation frequency and fundamental and harmonic amplitudes of phase shift oscillators, specifically those employing *RC* phase shift networks, although the method is quite general in its

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application. A novel extension to the method of *harmonic balance* to include harmonics in addition to the fundamental, as is normal practice, is proposed. This is significant because the method allows a more accurate calculation of the oscillation frequency, fundamental and harmonic amplitudes in oscillators with higher signal levels, less selective phase shift networks and/or higher levels of nonlinearity than is possible with existing methods. New closed form expressions for these parameters are presented enabling a more precise and effective design process for this type of oscillator.

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APPENDIX

Fourier Methods

A1. Basic principles

In order to develop the extended harmonic balance equations we require the Fourier series expansion of the general 3rd order polynomial expression for the output current from the nonlinear block as given by eqn (24):

$$I = G_0\underline{A} + \beta|\underline{A}|\underline{A} + \alpha\underline{A}^3 = I_1(t) + I_2(t) + I_3(t)$$

which, as indicated, is the linear superposition of contributions from the linear, quadratic and cubic terms. As already noted, for the forms of nonlinearity considered in this paper, the resulting Fourier series expansion consists of odd-order sin terms only, the even-order terms (including the dc term) being zero [15]. For example, referring to eqn (6) in the text, for the quadratic nonlinearity, the first three Fourier coefficients C_i are given by:

$$C_1 = \frac{8}{\pi}.\frac{1}{1.1.3} = \frac{8}{3\pi} = 0.85$$

$$C_3 = \frac{8}{\pi}.\frac{1}{1.3.5} = \frac{8}{15\pi} = 0.17$$

$$C_5 = \frac{8}{\pi}.\frac{1}{3.5.7} = \frac{8}{105\pi} = 0.024$$

$$\vdots$$

This observation allows the periodic voltage variable $\underline{A}$ to be represented as follows, truncated to two terms:

$$\underline{A} = A_1 \sin\omega t + A_3 \sin(3\omega t + \phi) \quad \text{(A1)}$$

where A_1 and A_2 are the amplitudes of the harmonic components and ϕ is the phase angle between them. Note that the third harmonic term $\underline{A}_3 = A_3 \sin(3\omega t + \phi)$ can be written in complex form:

$$\underline{A}_3 = A_3 \sin 3\omega t.(\cos\phi + j\sin\phi) \quad \text{(A2)}$$

A2. Extension to include the third harmonic

In this section the response of a nonlinear conductance to a weighted linear sum of fundamental sinusoid and its third harmonic is considered. The harmonic components at the input and output of the nonliearity are permitted to have arbitrary amplitude and phase relationships to the fundamental. Begin by considering the cubic term $I_3(t)$:

$$I_3(t) = \alpha\underline{A}^3$$

using (A1) the factor $\underline{A}^3$ can be further expanded into a polynomial consisting of the sum of four terms:

$$\underline{A}^3 = A_1^3 \sin^3 \omega t + 3A_1^2 A_3 \sin^2 \omega t + 3A_1 A_3^2 \sin\omega t \sin^2(3\omega t + \phi) + A_3^3 \sin^3(3\omega t + \phi)$$

Each term can now be expanded separately as the sum of a Fourier sine and a cosine series and finally the complete set summed and truncated to two terms as required. The analysis is routine and is not given in detail. The complete truncated output is written as the following complex function:

$$\underline{A}^3 = [(0.75A_1^3 - 0.75A_1^2 A_3) - j0.75A_1^2 A_3 \sin\varphi]\sin\omega t + [(-0.25A_1^3 + 1.50A_1^2 A_3 \cos\varphi) + j1.50A_1^2 A_3 \sin\varphi]\sin 3\omega t \quad \text{(A3)}$$

If we make the assumption that $A_1 >> A_3$, the second term in the real part of the expression at the fundamental frequency can be omitted making eqn (A3) entirely consistent with eqn (9b). This approximation is useful since it results in A_3 being eliminated from this equation, which allows A_1 to be calculated from the first equation of the pair (9b) as the first stage of the solution process.

It has already been demonstrated that this approximation does not result in a significant loss of accuracy. For the quadratic term in eqn (24), the analysis is somewhat more involved. This is because the product:

$$|A|A = |A\sin\omega t|A\sin\omega t$$

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is in general $\neq A^2 \sin^2 \omega t$, its actual value depending on which half-cycle of the driving sinusoid is in operation, i.e:

$$\left|A\sin\omega t\right|A\sin\omega t = A^2\sin^2\omega t;\ 0 \le t \le n\pi/\omega$$

$$-\ A^2\sin^2\omega t;\ n\pi/\omega \le t \le 2n\pi/\omega$$

where n is an integer. The half cycles must therefore be treated separately and the results summed. Hence the added difficulty referred to. The resulting complex function is as follows (after making similar approximations to those made in the cubic case):

$$f(t) = [0.85A_1^2 - j0.51A_1A_3\sin\varphi]\sin\omega t$$
$$+[(-0.17A_1^2 + 1.31A_1A_3\cos\varphi) + j1.24A_1A_3\sin\varphi]\sin 3\omega t \quad \text{(A4)}$$

which is consistent with eqn (9a).

Table I: Complex Transfer Functions for the *Wien* Bandpass Phase Shift Network at Resonance

Harmonic No.	<u>H</u>	<u>J</u>
1	1/3 + *j*0	1/2 + *j*0
3	0.19 - *j*0.16	0.18 - *j*0.24
5	0.094 - *j*0.15	0.073 – *j*0.18

Table II: Comparison of PSPICE simulations with simple calculations for a *Wien* bridge phase shift oscillator with a *quadratic* nonlinearity
(C = 120 pF, R = 50kΩ, β = 5e-4 AV^{-2}, R_1 = 1 kΩ)

	G_0 = 4.575e-4		G_0 = 0	
	PSPICE	Calcn 1	PSPICE	Calcn 1
E_1 (V)	0.05	0.05	0.61	0.59
E_3 (V)	6.2e-4	8.5e-4	0.066	0.20
E_3/E_1 (%)	1.2	1.7	11	34
f_0, kHz	26.51	26.52	24.94	26.52

Table III: Comparison of PSPICE simulations with improved calculations for a *Wien* bridge phase shift oscillator with a *quadratic* nonlinearity
(C = 120 pF, R = 50kΩ, β = 5e-4 AV^{-2}, R_1 = 1 kΩ)

	G_0 = 0		G_0 = 2.5e-4		G_0 = 4.575e-4	
Harmonic	PSPICE	Calcn 2	PSPICE	Calcn 2	PSPICE	Calcn 2
A_1 (volts)	1.23	1.23	0.61	0.59	0.10	0.10
A_3 (volts)	0.21	0.19	0.060	0.059	2.0e-3	2.02e-3
A_3/A_1 %	17	19	9.8	10	2	2
φ^o	30	24	33	29	37	35
f_0, kHz	24.94	25.48	26.06	26.15	26.51	26.52

Table IV: Comparison of PSPICE simulations with simple calculations for a *Highpass ladder* phase shift oscillator with a *quadratic* nonlinearity
(C = 120 pF, R = 50kΩ, β = 5e-4 AV^{-2})

	G_0 = 0, R_1 = 1kΩ		G_0 = 2.5e-4, R_1 = 100Ω	
Harmonic	PSPICE	Calcn 2	PSPICE	Calcn 2
A_1 (volts)	8.04e-2	8.40e-2	2.49e-1	2.55e-1
A_3 (volts)	2.75e-3	2.79e-3	2.80e-3	2.63e-3
A_3/A_1 %	3	3	1	1
φ^o	97	87	83	80
f_0, kHz	10.60	10.63	10.83	10.83

AUTHOR'S PROFILE

John Taylor
was born at Wanstead, Essex, UK. He received the BSc and PhD degrees from Imperial College, London University, UK, in 1973 and 1984, respectively. During 1984-1985, he held the post of Research Fellow in the Department of Electrical Engineering, University of Edinburgh, Scotland, where worked on switched-capacitor filter design. He joined the Department of Electronic and Electrical Engineering at University College London in 1985 and subsequently, in 2002, the Department of Electronic and Electrical Engineering at the University of Bath, Bath, UK, where he holds the position of Professor of Microelectronics and Optoelectronics and Director of the University *Centre for Advanced Sensor Technolgies*. His research interests are in the fields of analogue and mixed analogue and digital system design, including communication systems and low-power implantable systems for biomedical and rehabilitation applications. Professor Taylor has published more than 180 technical papers in international journals and conferences and has co-edited a handbook on filter design.